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On Numerical Solution of Kinetic Equations for Plasma Waves

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The main properties of a kinetic wave equation in isothermal plasma $T_e \simeq T_i$ are described. A fast method is suggested for numerical solution of kinetic equations for plasma waves. The method error is estimated. Results of solution of the problem on plasma heating with a relativistic electron beam are presented as an illustration of the method.

Plasma turbulence, i.e., the state of plasma characterized by a high level of variations in density and in electric and magnetic fields, has been actively investigated in the last few years. This is primarily because in the majority of cases of plasma heating, e.g., with the help of relativistic electron beam or laser pulse, the energy transforms first into collective wave motions. The characteristics of heating, such as relaxation length of the beam, energy flows into plasma, etc., are defined, thus, by nonlinear interaction of oscillations. The study of this range of questions began in 1961–1963, when similarly to kinetic effects in solid state, an description of the weak plasma turbulence according to the kinetic equations for waves [1, 2], was formulated. The subsequent investigations showed, however, that plasma turbulence behavior differs sharply from the quasi-particle kinetics in solid bodies. In the majority of cases the weak turbulence spectra have proved to be unsteady, concentrated in the **k**-space on the surfaces, lines or even series of points [3–8]. These important results have been obtained because of the wide use of numerical experiments. This paper presents the technique for numerical solution of the kinetic equation for Langmuir waves, which has proved to be effective when the temperatures of electrons and ions in plasma are comparable: $T_e \simeq T_i$. This case can be frequently met in practice.

In Section 1 some common properties of the kinetic equation for Langmuir waves are examined. In particular, it is shown that the equation is a Hamiltonian one, conserving phase volume, even in the presence of the source and wave attenuation. This fact defines, to a great extent, the dynamics of Langmuir turbulence. Section 2 presents the method for quick solution of a kinetic equation, based on approximate factorization of the kernel of integral equation. The error of such a transformation is

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evaluated; the error of solution is shown to be proportional to the kernel variation and can be rather small. Section 3 illustrates the results of solving the problem on the wave spectra for the plasma heated by a relativistic electron beam.

1. The Kinetic Wave Equation and Its Main Properties

The kinetic equation describing evolution of the spectrum of Langmuir turbulence due to induced ion scattering has the form [2, 4, 9]

$$\frac{\partial \varepsilon_k}{\partial t} = \varepsilon_k \left[\gamma_k + \int T_{kk'} \varepsilon_{k'} \, dk' \right] + f_k,$$

$$T_{kk'} = -T_{k'k} = \frac{\omega_p}{2nTe} \frac{(\mathbf{kk'})^2}{k^2 k'^2} \operatorname{Im} G_{\mathbf{k}-\mathbf{k}', \omega_k - \omega_{k'}}, \tag{1.1}$$

$$G_{\kappa\Omega} = \frac{L_{\kappa\Omega}}{1 - L_{\kappa\Omega}}; \qquad L_{\kappa\Omega} = \frac{T_e}{Mn} \int \frac{\boldsymbol{\kappa}(\partial f / \partial \mathbf{v})}{\boldsymbol{\kappa}\mathbf{v} - \Omega} \, d\mathbf{v}.$$

Here ε_k is the density of energy of plasma oscillations within the range of the wave vectors $\mathbf{k}$ and $\mathbf{k} + d\mathbf{k}$, $f(\mathbf{v})$ is the ion velocity distribution, and f_k is the source of thermal noise. The term $\gamma_k \varepsilon_k$ corresponds to linear attenuation and excitation of oscillations, ω_k is the Langmuir wave dispersion law, and ω_p is plasma frequency. Equations (1.1) describe nonlinear energy pumping in the $\mathbf{k}$-space from the area of wave excitation to the area of small wave vectors, where it dissipates. Equations of the (1.1) type are used in various fields of physics, for example, in the study of scattering of superthermal quasar radiation [10] or Mandelstam–Brillouin multiple scattering.

1.1. *Hamiltonian Nature and the Saturation Time Stationary Solutions*

Let us show at first that Eq. (1.1) is (if thermal noise is neglected) a Hamiltonian one which conserves the phase volume and other integral invariants. By definition $\varepsilon_k > 0$. Introduce now a new variable, $Z_k = \ln \varepsilon_k$. Equation (1.1) can be rewritten as

$$\int R_{kk'} \frac{\partial Z_{k'}}{\partial t} \, d\mathbf{k}' + [\tilde{\Gamma}_k - e_{Z_k}] = 0,$$

$$\tilde{\Gamma}_k = \int R_{kk'} \gamma_{k'} \, d\mathbf{k}', \tag{1.2}$$

where $R_{kk'}$ is a kernel of the operator which is inverse to an operator with the kernel $T_{kk'}$. It is obvious that $R_{kk'} = -R_{k'k}$.

It is easily seen that (1.2) can be obtained through varying the functional

$$S = \frac{1}{2}\int R_{kk'} Z_k \frac{\partial Z_{k'}}{\partial t}\, d\mathbf{k}\, d\mathbf{k}'\, dt + \int H\, dt,$$
$$H = \int [\tilde{\Gamma}_k Z_k + e^{Z_k}]\, d\mathbf{k}, \tag{1.3}$$

being an action for (1.2). Let us rewrite (1.2) in the form

$$\int R_{kk'} \frac{\partial Z_{k'}}{\partial t}\, d\mathbf{k}' = \frac{\delta H}{\delta Z_k}$$

or (1.4)

$$\frac{\partial Z_k}{\partial t} = \int T_{kk'} \frac{\delta H}{\delta Z_{k'}}\, d\mathbf{k}'$$

From (1.4) it follows that $\partial H/\partial t = 0$ due to the asymmetry $T_{kk'}$; i.e., $H = \int (\varepsilon_k + \tilde{\Gamma}_k \ln \varepsilon_k)\, d\mathbf{k}$ is the integral of motion for (1.4), or, equivalently, for (1.1). In the absence of damping and instabilities $\gamma_k = 0$, H transforms into a well-known integral of the total number of Langmuir waves I. The possibility of writing (1.1) in the form of (1.4) means that (1.1) is a Hamiltonian equation. From this it follows that Eq. (1.1) cannot have asymptotically stable stationary solutions with the finite saturation time unless f_k vanishes. Relaxation to the stationary state (saturation) only occurs if there is a source of thermal noise f_k, disturbing th Hamiltonian nature of (1.1) and giving it the character of a kinetic equation. Let us clarify dependence of the relaxation time upon noise level [7].

Let ε_k^0 be a stationary solution to (1.1):

$$\varepsilon_k^0 \left(\gamma_k + \int T_{kk'} \varepsilon_{k'}^0\, dk'\right) + f_k = 0. \tag{1.5}$$

Let us assume that $\varepsilon_k(t) = \varepsilon_k^0 + \delta\varepsilon_k(t)$ and

$$\delta\varepsilon_k \ll \varepsilon_k.$$

By linearization of (1.1) we obtain, owing to antisymmetry $T_{kk'}$, the relation

$$\frac{\partial}{\partial t}\int \frac{\delta\varepsilon_k^2}{\varepsilon_k^0}\, d\mathbf{k} = -\int f_k \frac{\delta\varepsilon_k^2}{\varepsilon_k^{02}}\, d\mathbf{k},$$

which provides evaluation of the relaxation time

$$\tau \simeq \varepsilon_{k\,\max}^0 / f_k. \tag{1.6}$$

1.2. *Stationary Solutions*

With the neglect of small noise terms, the stationary spectra ε_k^0 satisfy the equation

$$\varepsilon_k^0 \left(\gamma_k + \int T_{kk'} \varepsilon_{k'}^0 \, d\mathbf{k}' \right) = 0. \tag{1.7}$$

Besides, we can demand that the stationary state be stable in the areas of the **k**-space, where $\varepsilon_k^0 = 0$. This gives the relations

$$\gamma_k = \Gamma_k^{n1}, \qquad \varepsilon_k^0 \neq 0, \tag{1.8a}$$

$$\gamma_k < \Gamma_k^{n1}, \qquad \varepsilon_k^0 = 0, \tag{1.8b}$$

$$\Gamma_k^{n1} = -\int T_{kk'} \varepsilon_{k'}^0 \, d\mathbf{k}'. \tag{1.8c}$$

Equations (1.8) define the stationary solution and the area in the **k**-space, where $\varepsilon_k^0 \neq 0$ [4, 11].

Conditions (1.8) have simple geometrical sense. Let us consider them with a fixed module of the wave vector. Then the value Γ_k^{n1} and γ_k are the function of solid angle and relation (1.8) means that the surface γ_k is inside the surface Γ_k^{n1}. At the points where touching or "adhesion" of the surfaces occur, ε_k^0 is nonzero. Equation (1.8a) is a Fredholm equation of the first kind. Its solution is defined by the structure γ_k and the kernel $T_{kk'}$, and there is a great variety of them; some examples of stationary distributions ε_k^0 are given in Section 3.

Let us emphasize an important fact. It follows from (1.8) that Γ_k^{n1} breaks at those points where ε_k^0 is near zero. Since the kernel $T_{kk'}$ is analytical, the breaks, as one would think, are the result of the irregularity ε_k^0. But with ε_k^0 near zero we must use exact Eq. (1.5). The inclusion of the thermal noise results in a peculiar regularization of (1.1) and smoothing of the stationary solutions ε_k^0. This is of particular importance when the characteristic size γ_k is much less than the kernel "width" $T_{kk'}$, because in this case the stationary solution is a set of equidistant sharp peaks [5, 7].

2. Fast Method for Numerical Solution of a Kinetic Wave Equation

The standard method of numerical solution (1.1) consists in the use of a stable difference scheme over time, e.g., the Crank–Nicholson scheme, and replacement of integration by summation over a discrete set of points. But the computer time consumption is excessive even in the plane or axial symmetrical case: $(N_1 \times N_2)^2$ arithmetic operations (N_1 and N_2 are the number of points in the corresponding dimensions) must be performed at each time step.

The suggested method uses replacement of the kernel of integral equation (1.1) by a similar degenerated one. In so doing, as we shall see, calculation of the right-hand

side (1.1) "breaks up," and the calculation time decreases by more than an order of magnitude.

The next point of principle consist in the following: it was shown in Section 1 that the saturation time and existence itself of the stationary state are defined by the noise terms which disturb the Hamiltonian nature of (1.1).

Therefore variation of the kernel $T_{kk'}$ (the asymmetry properties remain) will result in the change of the amplitude of the stationary distributions ε_k^0 only.

For simplicity, we present the principle of the method and error evaluation for a one-dimensional case.

2.1. *Replacement of the Kernel*

Let us replace the kernel $T_{kk'}$ by a similar degenerated one,

$$\tilde{T}_{kk'} = \sum_{i=0}^{n} a_i(k)\, b_i(k'). \tag{2.1}$$

The Taylor series can be taken as (2.1), but it is suitable only in the vicinity of a chosen point. Our purpose is to approximate $T_{kk'}$ through the whole interval of definition, but not at the individual points. Therefore it is natural to replace the kernel by a finite number of n terms of an expansion into orthogonal polynomials. In our opinion, Tchebyshev polynomials should be preferred in this class: first, the error of expansion for them is uniform on the whole interval; second, and this is of particular importance in our case, the series in the Tchebyshev polynomials converges rapidly, and it can be broken when n are small [12]. So,

$$\tilde{T}_{kk'} = \sum_{i=0}^{n} a_i(k)\, T_i(k') \tag{2.2}$$

(it is supposed that the interval over k has been already reduced to a unit interval $(-1, 1)$). Then

$$\Gamma_k^{n1} = \int T_{kk'} \varepsilon_{k'}\, d\mathbf{k}' \simeq \sum_{i=0}^{n} a_i(k)\, M_i, \tag{2.3}$$

where $M_i = \int \varepsilon_{k'} T_i(k')\, d\mathbf{k}'$ are the moments $\varepsilon_{k'}$ in the Tchebyshev polynomials. At each time step $t = j\tau$ the moments M_i of $\varepsilon_{k'}$ $(t = j\tau)$ should be calculated first, and then Γ_k^{n1} with the help of (2.3). A criterion for the choise of the n-number of the series terms in specific calculations can be as follows: the discrepancy of solution (1.1) should be calculated as n increases, and compared with the error of the difference time scheme. The numerical experiment has shown that for the schemes of the Crank–Nicholson type of the second order of accuracy and $\tau \simeq (1 \div 5) \times 10^{-2}$ the corresponding n is of the order of $5 \div 7$.

2.2. *Estimation of Error*

Let us rewrite Eq. (1.8) for the stationary distribution ε_k^0 as

$$\int K(x, y)\, \varphi(y)\, dy = f(x). \tag{2.4}$$

If the kernel $K(x, y)$ is degenerate, i.e.,

$$K(x, y) \Rightarrow \tilde{K}(x, y) = \sum_{i=0}^{n} a_i(x)\, b_i(y)$$

solution (2.4) will be [13]

$$\varphi(x) = \int f(y)\, R(x, y)\, dy, \tag{2.5}$$

where

$$R(x, y) = \frac{1}{\Delta} \sum_{mj} \tilde{M}_{jm} b_m(y)\, a_j(x),$$

$$\Delta = \mathrm{Det}\{M_{jm}\},$$

$$M_{jm} = \sum_{s}^{n} b_{sj} b_{sm},$$

$$b_{sj} = \int b_s(y)\, a_j(y)\, dy.$$

$\tilde{M}_{jm}$ is an algebraic cofactor of the element of the jth column and mth line of the matrix M_{jm}. Let the difference between the real kernel and the similar factorized kernel be of the order of δ

$$\int |K(x, y) - \tilde{K}(x, y)|\, dy < \delta \tag{2.6}$$

through the whole interval over x. Rewrite (2.4) in the form

$$\int \tilde{K}(x, y)\, \varphi(y)\, dy + \int (K(x, y) - \tilde{K}(x, y))\, \varphi(y)\, dy = f(x)$$

or

$$\int \tilde{K}(x, y)\, \varphi(y)\, dy = \tilde{f}(x),$$

$$\tilde{f}(x) = f(x) + \int (\tilde{K}(x, y) - K(x, y))\, \varphi(y)\, dy. \tag{2.7}$$

With the help of (2.7) we obtain the estimation of $\varphi_{\max}$

$$|\tilde{f}| \leqslant |f| + \left| \int (\tilde{K}(x, y) - K(x, y))\, \varphi(y)\, dy \right|$$

$$\leqslant |f|_{\max} + \varphi_{\max} \int |\tilde{K}(x, y) - K(x, y)|\, dy$$

$$\leqslant |f|_{\max} + \varphi_{\max} \delta,$$

$$|f|_{\max} = \sup_x |f(x)|; \qquad \varphi_{\max} = \sup_x \varphi(x).$$

Note that (2.7) is again an equation with degenerated kernel, and its solution has the form (see (2.5))

$$\varphi(y) = \int R(x, y)\, \tilde{f}(y)\, dy.$$

This suggests that $\varphi_{\max} \leqslant R_{\max} |\tilde{f}| < R_{\max}(|f|_{\max} + \varphi_{\max} \delta)$,

$$\mathrm{R}_{\max} = \sup \int |R(x, y)|\, dy. \tag{2.8}$$

And, finally, $\varphi_{\max} < |f|_{\max} R_{\max}$ for rather small δ ($R_{\max} \delta \ll 1$).

Now we write down Eq. (2.4) for difference of the solutions with exact and reduced kernels $\delta\varphi = \varphi - \tilde{\varphi}$:

$$\int \delta\varphi(y)\, \tilde{K}(x, y)\, dy = \tilde{f}(x) - f(x). \tag{2.9}$$

Because of (2.5) solution (2.9) is

$$\delta\varphi(x) = \int R(x, y)[\tilde{f}(y) - f(y)]\, dy.$$

Let us estimate the solution

$$|\delta\varphi| \leqslant R_{\max}\, \delta\varphi_{\max} < R_{\max}^2\, \delta\, |f|_{\max}.$$

Thus, the stationary solution error is proportional to the magnitude of kernel variation and can be infinitesimal.

3. Langmuir Wave Spectra in the Problem of Plasma Heating with a Relativistic Beam

The beam of relativistic electrons in plasma, where electron and ion temperatures are comparable, excites Langmuir oscillations located in the **k**-space near the surface

of resonance of plasma particles with the beam

$$\omega_p = \mathbf{k}\mathbf{v}_b. \tag{3.1}$$

The increment of beam instability γ_k is nonzero for $k > \omega_p/v_b$ and reaches its maximum, $\gamma_{\max}$, when $k = \omega_p/v_b$.

The value $\gamma_{\max} \simeq \eta\omega_p(1/(\Delta\Theta)^2)(v_b^2/c^2)$, where $\Delta\Theta$ is a value angular velocity dispersion of the beam particles, η is a ratio of particle densities in a beam and plasma, and v_b is a beam velocity [14]. Characteristic time of variation of electron velocity distribution function f_e is much more than characteristic time of nonlinear wave interaction. Therefore, at first we can solve a problem about Langmuir turbulence spectra resulting from the induced scattering on ions of Langmuir waves for the given $f_e(v)$, and then take into account the inverse action of oscillations on the distribution function.

Let us write (1.1) for a case of axial symmetry, which corresponds to a typical experiment, and consider the initial value problem for Eq. (1.1). We assume that at time $t = 0$ an electron beam enters plasma only with a small thermal noise. As the results of numerical experiment show, stationary wave distribution is practically independent of detailed structure of the initial distribution ε_k $(t = 0)$, and for the sake of simplicity we can assume that ε_k $(t = 0) = f = \text{const}$.

Let us average $T_{kk'}$, i.e., a kernel of integral equation (1.1), with a Maxwell function of ion velocity distribution in the azimuth angle φ, after which it remains a function k, k' and Θ, Θ'. It is convenient to change from the variable of angle Θ to $x = \cos\Theta$. According to the procedure described in Sections 1 and 2 further we should factorize the kernel $T(kk', xx')$ in k', x'. A more detailed analysis shows, however, that in the problem considered the $\varepsilon_{k,\Theta}$ wave distribution over the module k at fixed $x = \cos\Theta$ is heavily cut up (see Fig. 1), and a significant gain in the computation time can be attained for splitting in the variable x only.

An implicit differential scheme of the second time order of the Crank–Nicholson type was applied to the factorized equation

$$\frac{\varepsilon^{i+1} - \varepsilon^{i-1}}{2\tau} = \frac{\varepsilon^{i+1} + \varepsilon^{i-1}}{2}\,[\gamma + \Gamma(\varepsilon^i)], \tag{3.2}$$

where τ is a step of the time discretization $\varepsilon^i = \varepsilon_{kx}$ $(t = i\tau)$. For typical versions the number of points over module k was equal to $100 \div 120$, over angle $\Theta - 32$. At first the oscillations increased near the surface (3.1), then there occurred nonlinear energy pumping from the area of excitation into nonresonance area $k < \omega_p/v_b$ because of the induced ion scattering. The numerical experiment has shown that with $k < \omega_p/v_b$, the oscillations are concentrated in narrow areas, i.e., "jets," $\Theta < 15°$, $\pi - \Theta < 15°$ (see Fig. 1), the angle Θ is counted off the direction of beam propagation. The length of the jets is defined by the ratio of $\gamma_{\max}$ to the attenuation of waves ν_{ei} caused by electron collisions. The characteristic angular width of the jets increases with increasing angular dispersion $\Delta\Theta$ of the beam particles, but still it is much less than

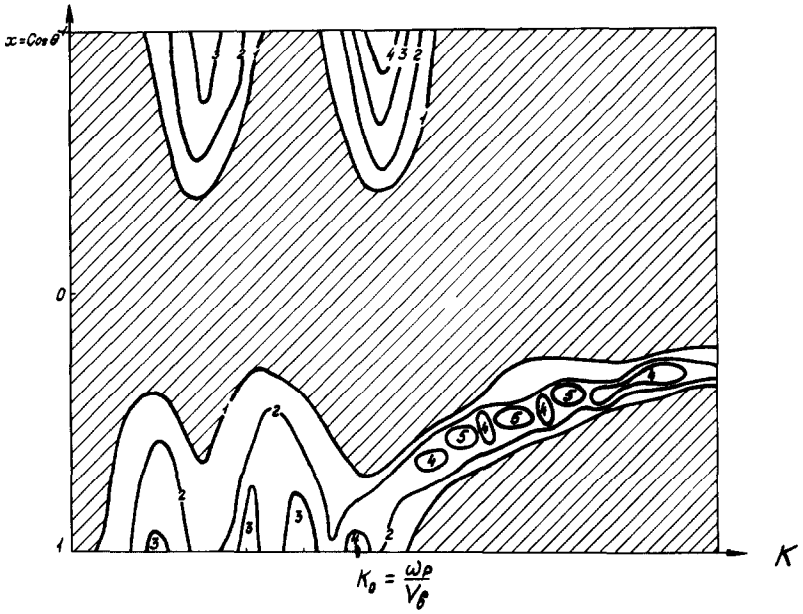

FIG. 1. The curves of constant value of the function $\ln(\varepsilon_k/\varepsilon_{k\,\text{noise}})$.

$\pi/2$; i.e., the ε_k distributions are anisotropic, even when the source of oscillations differs slightly from an isotropic one. The structure of the jets over k is defined by the ratio of the excitation area (the size at which γ_k changes substantially) to the characteristic scale of wave interaction, i.e., to the kernel "width" $\Delta k = k_{\text{dif}} = (\omega_p/v_{T_e})\sqrt{\mu}$, and it can be strongly diverse (for more details see [4, 7]).

Here v_{T_e} is thermal velocity of electrons, μ is the ratio of electron mass to ion mass.

Oscillations in the $k > \omega_p/v_b$ area are concentrated on the two-dimensional "stream," whose position coincides with the surface of maximum increment (3.1) to an accuracy of $\Delta\Theta$. The distribution of the oscillations along the stream looks like a plateau with sharp peaks spaces at the k_{dif} distance.

The dynamics of arrangement of the stationary wave spectra has also proved to be rather uncommon. As the numerical experiment has shown the process of arrangement has two stages. At the first stage there occurs rough arrangement of the ε_k spectrum. The main characteristics of the system of waves, e.g., the total energy of oscillations $E = \int \varepsilon_k \, d\mathbf{k}$, are near the stationary values in this case. The duration of this stage is defined by the thermal noise level

$$\tau_1 \sim \nu_{ei}^{-1} \ln \frac{\varepsilon_{k\,\max}^0}{\varepsilon_{k\,\text{noise}}}.$$

The second stage is much longer; according to estimation (1.6) which is confirmed

well by the numerical calculation

$$\tau_2 \sim \gamma_{\max}^{-1} \frac{\varepsilon_{k\,\max}}{\varepsilon_{k\,\text{noise}}} \gg \tau_1$$

(for the typical plasma parameters $\tau_2 \simeq (100 + 200)\,\gamma_{\max}^{-1}$ $\tau_1 \sim 10\gamma_{\max}^{-1}$). The fine structure of the wave spectra arranges finally during this period. The computer time consumption for calculation of one version up to $t \sim \tau_2$ according to the technique described is $1.5 + 2$ hr using the computer BESM-6.

The results of this problem solution (anisotropy of spectra, dynamics of the stationary distribution arrangement) allow a number of conclusions of principle about the mechanism of the beam–plasma interaction, in particular, the conclusion on the three-dimensional nature of relativistic electron beam relaxation, resulting in considerable broadening of the velocity distribution function of particles. It also allows calculation of the relaxation length, which is the main characteristic of plasma heating.

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